ON STRESS CONCENTRATIONS IN THICK PLATES

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Investigation is continued on the applicability of applied plate bending theory, based on the Kirchhoff hypothesis, to the solution of stress concentration problems by means of the method developed in [1 and 2].

1. Consider the problem of axisymmetric bending of an infinite plate of a thickness $2h$ with an opening which is bounded by the circular cylindrical surface Γ with radius a. Introduce dimensionless coordinates s, n and ζ (Fig. 1). Assume that the flat faces of the plate are traction-free while the cylindrical surface is subjected to a normal load $N=k\lambda\zeta^m$, and is free of shear. Here, m is an odd integer, k is a constant of proportionality and $\lambda = h/a$.

Applying the method described in [1 and 2], we obtain expressions for the stresses on the boundary Γ, which are of the form

$$\sigma_n|_\Gamma = 2\mu\lambda\left\{\left[2\nu\frac{\partial^2\psi_0}{\partial n^2}+(\nu-1)\frac{\partial\psi_0}{\partial n}\right]_{n=0}\zeta+\sum_{p=1}^{\infty}[(\nu-1)s_p(\zeta)+\gamma_p^2 n_p(\zeta)]c_{p2}\right\}+$$
$$+2\mu\lambda^2\left\{\left[2\nu\frac{\partial^2\psi_1}{\partial n^2}+(\nu-1)\frac{\partial\psi_1}{\partial n}\right]_{n=0}\zeta+\sum_{p=1}^{\infty}[(\nu-1)s_p(\zeta)+\gamma_p^2 n_p(\zeta)]c_{p3}+\right.$$
$$\left.+\sum_{p=1}^{\infty}\gamma_p n_p(\zeta)c_{p2}\right\}+2\mu\lambda^3\left\{\left[2\nu\frac{\partial^2\psi_2}{\partial n^2}+(\nu-1)\frac{\partial\psi_2}{\partial n}\right]_{n=0}\zeta-\right.$$
$$-\frac{1}{2}\left(\nu+\frac{1}{3}\right)\frac{\partial^2\Delta\psi_0}{\partial n^2}\Big|_{n=0}\zeta^3+\sum_{p=1}^{\infty}[(\nu-1)s_p(\zeta)+\gamma_p^2 n_p(\zeta)]c_{p4}+$$
$$\left.+\sum_{p=1}^{\infty}n_p(\zeta)\left(\gamma_p c_{p3}+\frac{1}{2}c_{p2}\right)\right\}+\ldots \tag{1.1}$$

$$\sigma_s|_\Gamma = 2\mu\lambda\left\{\left[2\nu\frac{\partial\psi_0}{\partial n}+(\nu-1)\frac{\partial^2\psi_0}{\partial n^2}\right]_{n=0}\zeta+(\nu-1)\sum_{p=1}^{\infty}s_p(\zeta)c_{p2}\right\}+$$
$$+2\mu\lambda^2\left\{\left[2\nu\frac{\partial\psi_1}{\partial n}+(\nu-1)\frac{\partial^2\psi_1}{\partial n^2}\right]_{n=0}\zeta+\sum_{p=1}^{\infty}[(\nu-1)s_p(\zeta)c_{p3}-\gamma_p n_p(\zeta)c_{p2}]\right\}+ \tag{1.2}$$

$$+2\mu\lambda^3\left\{\left[2\nu\frac{\partial\psi_2}{\partial n}+(\nu-1)\frac{\partial^2\psi_2}{\partial n^2}\right]_{n=0}\zeta-\frac{1}{2}\left(\nu+\frac{1}{3}\right)\frac{\partial\Delta\psi_0}{\partial n}\bigg|_{n=0}\zeta+\right.$$

$$\left.+\sum_{p=1}^{\infty}\left[(\nu-1)\,s_p(\zeta)\,c_{p4}-\gamma_p n_p(\zeta)\,c_{p3}-\frac{1}{2}\,n_p(\zeta)\,c_{p2}\right]\right\}+\ldots$$

$$\tau_{nz}|_{\Gamma}=-2\mu\lambda\sum_{p=1}^{\infty}\gamma_p r_p(\zeta)\,c_{p2}+2\mu\lambda^2\left\{\nu(1-\zeta^2)\frac{\partial\Delta\psi_0}{\partial n}\bigg|_{n=0}-\right.$$

$$\left.-\sum_{p=1}^{\infty}r_p(\zeta)\left(\gamma_p c_{p3}+\frac{1}{2}c_{p2}\right)\right\}+\ldots \qquad (1.3)$$

$$\sigma_{z\,\Gamma}=2\mu\lambda\sum_{p=1}^{\infty}t_p(\zeta)\,c_{p2}+2\mu\lambda^2\sum_{p=1}^{\infty}t_p(\zeta)\,c_{p3}+\ldots \qquad (1.4)$$

$$\tau_{ns}=0,\qquad \tau_{sz}=0,\qquad \nu=\frac{1}{1-2\sigma} \qquad (1.5)$$

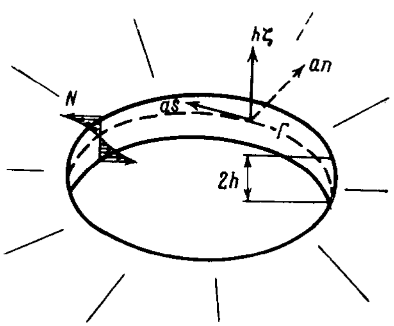

FIG. 1

Here μ is the shear modulus, σ is Poisson's ratio (in calculations $\sigma = 1/3$); $n_p(\zeta)$, $r_p(\zeta)$, $s_p(\zeta)$, and $t_p(\zeta)$ are known functions of ζ given in [1 and 3], $2\gamma_p$ are the roots of function $x^{-1}\sin(x) - 1$. Summation is carried out over the roots which have a positive real part.

The first 80 of these roots were obtained with aid of an electronic computer. Table 1 contains the values of forty roots which are located in the first quadrant. The roots in the fourth quadrant are conjugates of the above. The roots are numbered in the order of increasing magnitude, the odd-numbered roots being those in the first quadrant while the even-numbered roots are in the fourth quadrant.

In (1.1) to (1.4), the quantities $\psi_j(n)$ are biharmonic functions for the exterior of a circle, with $\psi_0(n)$ being the solution for the bending of an infinite plate with a circular opening as given by the applied theory. The boundary values of the functions $\psi_j(n)$ and the constants c_{pj} are obtained from the boundary conditions on Γ by means of an infinite system of equations.

The boundary conditions for $\psi_0(n)$ are given by

$$\frac{1}{3}\left[2\nu\frac{\partial^2\psi_0}{\partial n^2}+(\nu-1)\frac{\partial\psi_0}{\partial n}\right]_{n=0}=\frac{k}{\mu}\,\frac{1}{2(m+2)},\qquad \frac{\partial\Delta\psi_0}{\partial n}\bigg|_{n=0}=0 \qquad (1.6)$$

It is readily seen that the above coincide with the conditions of the applied plate bending theory based on the Kirchhoff hypothesis.

From (1.6), we have

$$\psi_0=B_0-\frac{k}{\mu}\,\frac{3}{2(\nu+1)(m+2)}\ln(n+1) \qquad (1.7)$$

TABLE 1

p	1	3	5	7	9
$\operatorname{Re}\gamma_p$	3.7488381	6.9499798	10.119259	13.277274	16.429870
$\operatorname{Im}\gamma_p$	1.3843390	1.6761046	1.8583834	1.9915704	2.0966252
p	11	13	15	17	19
$\operatorname{Re}\gamma_p$	19.579408	22.727036	25.873384	29.018831	32.163617
$\operatorname{Im}\gamma_p$	2.1833970	2.2573196	2.3217134	2.3787569	2.4299576
p	21	23	25	27	29
$\operatorname{Re}\gamma_p$	35.307902	38.451800	41.595390	44.738731	47.881869
$\operatorname{Im}\gamma_p$	2.4764020	2.5188989	2.5580670	2.5943901	2.6282535
p	31	33	35	37	39
$\operatorname{Re}\gamma_p$	51.024838	54.167664	57.310371	60.452973	63.595487
$\operatorname{Im}\gamma_p$	2.6599693	2.6897936	2.7179394	2.7445856	2.7698838
p	41	43	45	47	49
$\operatorname{Re}\gamma_p$	66.737923	69.880291	73.022600	76.164856	79.307064
$\operatorname{Im}\gamma_p$	2.7939639	2.8169378	2.8389026	2.8599433	2.8801345
p	51	53	55	57	59
$\operatorname{Re}\gamma_p$	82.449231	85.591359	88.733453	91.875516	95.017552
$\operatorname{Im}\gamma_p$	2.8995421	2.9182247	2.9362345	2.9536182	2.9704179
p	61	63	65	67	69
$\operatorname{Re}\gamma_p$	98.159562	101.30155	104.44351	107.58546	110.72739
$\operatorname{Im}\gamma_p$	2.9866716	3.0024137	3.0176753	3.0324849	3.0468686
p	71	73	75	77	79
$\operatorname{Re}\gamma_p$	113.86930	117.01119	120.15307	123.29494	126.43680
$\operatorname{Im}\gamma_p$	3.0608501	3.0744514	3.0876925	3.1005920	3.1131672

In order to exclude rigid body motion of the plate, we set $B_0 = 0$.

To determine c_{p2}, we have an infinite set of linear algebraic equations

$$\sum_{\substack{p=1\\p\neq t}}^{\infty} \frac{\gamma_p^2\gamma_t^2(\cos^2\gamma_t-\cos^2\gamma_p)}{(\gamma_t^2-\gamma_p^2)^2(\gamma_t-\gamma_p)}\left[\nu(\gamma_t+\gamma_p)^2-(\gamma_t-\gamma_p)^2\right]c_{p2}-\frac{\nu}{2}\gamma_t^3\left(\frac{2}{3}\cos^2\gamma_t-1\right)c_{t2}=F_{t2}$$

$$F_{t2}=\frac{k}{\mu}\left[\frac{\gamma_t}{8\nu}\int_{-1}^{1}\zeta^m n_t(\zeta)\,d\zeta+\frac{3(\nu-1)}{4\nu(m+2)}\frac{\sin^2\gamma_t}{\gamma_t}\right]\qquad(t=1,\ 2,\ 3,\ \ldots) \tag{1.8}$$

which may be written in the form

$$\|M\|\cdot\|c_{t2}\|=\|F_{t2}\| \tag{1.9}$$

Here $\|M\|$ is the complex matrix of the left-hand side of (1.8) and $\|c_{t2}\|$ and $\|F_{t2}\|$

are, respectively, the column matrix of the unknowns and that of the right-hand side of (1.8).

To transform (1.8) into real form, set

$$c_{p2} = u_{p2} - iv_{p2} \tag{1.10}$$

Noting that γ_{2n-1} is the conjugate of γ_{2n}, it can be easily shown that

$$u_{2n-1,2} = u_{2n,2}, \qquad v_{2n-1,2} = -v_{2n,2} \qquad (n = 1, 2, 3, \ldots) \tag{1.11}$$

so that the order of the real system is halved. If we limit ourselves to twenty boundary layers corresponding to first twenty roots γ_p lying in the right-hand side semi-plane, the order of the system will be equal to twenty. Introducing the notation

$$f_{2n-1,2} = \frac{\mu}{k}\,\mathrm{Re}\,F_{2n-1,2}, \qquad f_{2n,2} = \frac{\mu}{k}\,\mathrm{Im}\,F_{2n-1,2} \qquad (n = 1, 2, 3, \ldots) \tag{1.12}$$

$$x_{2n-1} = \frac{\mu}{k}\,u_{2n-1,2}, \qquad x_{2n} = \frac{\mu}{k}\,v_{2n-1,2} \tag{1.13}$$

we can write (1.9) in the form

$$\| M_1 \| \cdot \| x_j \| = \| f_{j2} \| \tag{1.14}$$

where $||M_1||$ is the matrix of the transformed real system.

System (1.14) is solved by truncation. With the aid of a computer, matrices of rank 20, 18, ... , 4 were successively inverted.

In [1], it was shown that, for a given material, the matrix is universal, i.e. it is independent of the loading or geometry of the plate, so that the results of the matrix inversion may be used for any plate bending problem. The inverted matrices permit the determination of the first fourteen out of the twenty unknown x_j, the accuracy of the approximation being insufficient for the remainder. Calculations were carried out for $m = 3$ and $m = 5$. The results are shown in Table 2.

TABLE 2

$m=3$	j	1	2	3	4	5
	$x_j=$	$0.180170\cdot10^{-2}$	$-0.12745\cdot10^{-3}$	$0.1993\cdot10^{-4}$	$-0.7332\cdot10^{-4}$	$-0.797\cdot10^{-5}$
	j	6	7	8	9	10
	$x_j=$	$-0.9967\cdot10^{-5}$	$-0.400\cdot10^{-5}$	$-0.119\cdot10^{-5}$	$-0.177\cdot10^{-5}$	$0.15\cdot10^{-6}$
	j	11	12	13	14	
	$x_j=$	$-0.83\cdot10^{-6}$	$0.30\cdot10^{-6}$	$-0.42\cdot10^{-6}$	$0.23\cdot10^{-6}$	
$m=5$	j	1	2	3	4	5
	$x_j=$	$0.18609\cdot10^{-2}$	$-0.689\cdot10^{-4}$	$0.1785\cdot10^{-3}$	$-0.1152\cdot10^{-3}$	$0.119\cdot10^{-4}$
	j	6	7	8	9	10
	$x_j=$	$-0.318\cdot10^{-4}$	$-0.18\cdot10^{-5}$	$-0.842\cdot10^{-5}$	$-0.20\cdot10^{-5}$	$-0.24\cdot10^{-5}$
	j	11	12	13	14	
	$x_j=$	$-0.12\cdot10^{-5}$	$-0.7\cdot10^{-6}$	$-0.7\cdot10^{-6}$	$-0.2\cdot10^{-6}$	

To illustrate the rate of convergence of the process used in the determination of x_j, we list various approximations of x_1, x_2, x_{13} and x_{14} as the most typical (Table 3). The superscript indicates the order the system from which the particular determination was made.

TABLE 3

	n	$x_1^{(n)}$	$x_2^{(n)}$	$x_{13}^{(n)}$	$x_{14}^{(n)}$
	6	$0.1803134\cdot10^{-2}$	$-0.128097\cdot10^{-3}$		
	8	$0.1801734\cdot10^{-2}$	$-0.127304\cdot10^{-3}$		
	10	$0.1801719\cdot10^{-2}$	$-0.127338\cdot10^{-3}$		
$m=3$	12	$0.1801708\cdot10^{-2}$	$-0.127381\cdot10^{-3}$		
	14	$0.1801705\cdot10^{-2}$	$-0.127410\cdot10^{-3}$		
	16	$0.1801703\cdot10^{-2}$	$-0.127429\cdot10^{-3}$	$-0.458\cdot10^{-6}$	$0.28\cdot10^{-6}$
	18	$0.1801702\cdot10^{-2}$	$-0.127439\cdot10^{-3}$	$-0.436\cdot10^{-6}$	$0.25\cdot10^{-6}$
	20	$0.1801702\cdot10^{-2}$	$-0.127443\cdot10^{-3}$	$-0.428\cdot10^{-6}$	$0.24\cdot10^{-6}$
	6	$0.187135\cdot10^{-2}$	$-0.73145\cdot10^{-4}$		
	8	$0.186330\cdot10^{-2}$	$-0.69875\cdot10^{-4}$		
	10	$0.186181\cdot10^{-2}$	$-0.68719\cdot10^{-4}$		
$m=5$	12	$0.186139\cdot10^{-2}$	$-0.68742\cdot10^{-4}$		
	14	$0.186119\cdot10^{-2}$	$-0.68802\cdot10^{-4}$		
	16	$0.186109\cdot10^{-2}$	$-0.68855\cdot10^{-4}$	$-0.128\cdot10^{-5}$	$-0.83\cdot10^{-7}$
	18	$0.186101\cdot10^{-2}$	$-0.68893\cdot10^{-4}$	$-0.74\cdot10^{-6}$	$-0.13\cdot10^{-6}$
	20	$0.186098\cdot10^{-2}$	$-0.68919\cdot10^{-4}$	$-0.70\cdot10^{-6}$	$-0.16\cdot10^{-6}$

We will see later that the accuracy with which the x_j were obtained is sufficient for practical purposes.

Utilizing (1.1) to (1.4), we may now obtain a first approximation of all components of the state of stress in the plate. These contain the infinite series of the boundary layer formulation. Let us examine the rate of convergence of these series at the most typical points. Here and hereinafter we denote the coefficients of λ^i in the series expressions for the stresses σ_n, σ_s and τ_{nz} on Γ by σ_{ni}, σ_{si} and τ_{nzi}, respectively.

Substituting (1.7), (1.10), (1.13) and the values previously obtained for x_j into (1.1) and (1.3), we obtain, for $m = 3$,

$$\sigma_{n_1}|_{\Gamma,\zeta=\pm1} = \pm k\{0.6000 + [49.3082 - 1.924 - 2.792 - 1.806 - - 1.102 - 0.69 - 0.45 - \ldots]\cdot10^{-2}\} \approx \pm k\,(0.6000 + 0.405) = \pm k\cdot1.005$$

$$\tau_{nz_1}|_{\Gamma,\zeta=0} = -k\,[7.4879 - 10.885 + 4.919 - 2.179 + 0.919 - 0.31 + 0.06 - \ldots]\cdot10^{-2} \approx \approx -k\cdot0.0001$$

For $m = 5$, we obtain

$$\sigma_{n_1}|_{\Gamma,\zeta=\pm1} = \pm k\{0.4286 + [52.492 + 13.056 - 0.044 - 1.80 - 1.68 - 1.28 - 0.9 - \ldots]\cdot10^{-2}\} \approx \pm k\,(0.4286 + 0.598) = \pm k1.027$$

$$\tau_{nz_1}|_{\Gamma,\zeta=0} = -k\,[5.386 - 11.684 + 10.42 - 6.63 + 4.0 - 2.5 + 1.4 - \ldots]\,10^{-2} \approx \approx -k0.004$$

The boundary conditions yield

$$\sigma_n|_{\Gamma,\zeta=\pm1} = \pm k\lambda, \qquad \tau_{nz}|_{\Gamma,\zeta=0} = 0 \tag{1.15}$$

Comparing the immediately preceding results with (1.15), we see that even at the points $\zeta = \pm 1$, where one would expect convergence to be the slowest, the series results

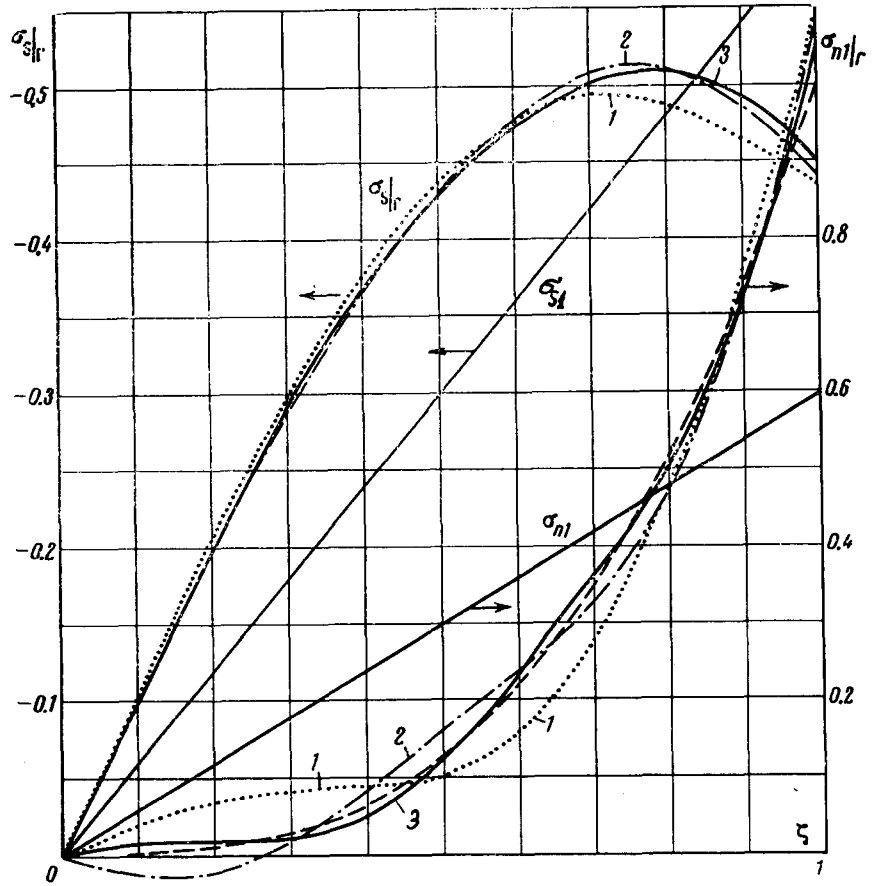

FIG. 2

using seven terms do not differ significantly from the exact ones. At other points, convergence is even better.

Fig. 2 shows curves of successive approximations for $\sigma_{n1}|_\Gamma(\zeta)$ using one, two and three boundary layer terms with $m = 3$. The straight lines in the figure correspond to the Kirchhoff solution; the broken line is the exact solution; curves *1*, *2* and *3* correspond to solutions taking into account one, two and three boundary layer terms, respectively.

Now let us calculate the first approximation of $\sigma_s|_\Gamma$ at the points $\zeta = \pm 1$. This stress is usually the basis for determining the stress concentration factor. $\sigma_s|_\Gamma$ may be calculated from formula (1.2), but it is easily shown that a first approximation of $\sigma_s|_{\Gamma,\ \zeta = \pm 1}$ for arbitrary m may be obtained without solving the infinite system for the determination of c_{p2}.

Substituting (1.7) into (1.1) and (1.2) and taking into account (1.11) as well as

$$(\nu - 1)\, s_p(\pm 1) + \gamma_p^2 n_p(\pm 1) = \pm 2\nu\gamma_p^2 = 2\nu s_p(\pm 1)$$

we obtain

$$\sigma_{n1}\,|_{\Gamma,\zeta=\pm1} = \pm k\left\{\frac{3}{m+2} + 8\nu\frac{\mu}{k}\sum_{p=1,\,3,\,\ldots}\operatorname{Re}(\gamma_p^2 c_{p_2})\right\}$$

$$\sigma_{s1}\,|_{\Gamma,\,\zeta=\pm1} = \pm k\left\{-\frac{3}{m+2} + 4(\nu-1)\frac{\mu}{k}\sum_{p=1,\,3,\,\ldots}\operatorname{Re}(\gamma_p^2 c_{p_2})\right\}$$

But

$$\sigma_{n1}\,|_{\Gamma,\zeta=\pm1} = \pm k$$

Hence

$$\sum_{p=1,\,3,\,\ldots}\operatorname{Re}(\gamma_p^2 c_{p_2}) = \frac{k}{8\mu\nu}\,\frac{m-1}{m+2}$$

Whereupon, we have

$$\sigma_{s1}\,|_{\Gamma,\zeta=\pm1} = \pm k\left\{-\frac{3}{m+2} + \frac{\nu-1}{2\nu}\,\frac{m-1}{m+2}\right\} \tag{1.16}$$

In (1.16), the first term in the braces corresponds to the solution of applied theory. From (1.16) it is clear that in this case, for $m \neq 1$ of course, the exact stress concentration factor is not obtained asymptotically from the Kirchhoff theory. The error of this theory, in the first approximation, increases as m increases. For $m = 3$, the error of applied theory is 22%; for $m = 5$ it is 44%; for $m = 7$ it is 66%, etc.

At other points on the surface Γ the error will also not be small. Fig. 2 shows the curves of the function $\sigma_{s1}|_\Gamma(\zeta)$, as calculated by means of the Kirchhoff theory (straight line) as well as those using one, two and three boundary layer terms (1, 2 and 3).

2. As discussed in [1], the next step in constructing the asymptotic expansion of the solution to the problem is the determination of $\psi_1(n)$ and c_{p3}.

The boundary conditions for $\psi_1(n)$ are given by

$$\frac{1}{3}\left[2\nu\frac{\partial^2\psi_1}{\partial n^2} + (\nu-1)\frac{\partial\psi_1}{\partial n}\right]_{n=0} = (\nu-1)\sum_{p=1}^{\infty}\frac{\sin^2\gamma_p}{\gamma_p}c_{p_2}, \qquad \frac{\partial\Delta\psi_1}{\partial n}\Big|_{n=0} = 0 \tag{2.1}$$

From (2.1), we have

$$\psi_1 = B_1 - 3\,\frac{\nu-1}{\nu+1}\sum_{p=1}^{\infty}\frac{\sin^2\gamma_p}{\gamma_p}c_{p_2}\ln(n+1) \tag{2.2}$$

As before, we set $B_1 = 0$. The system of equations for c_{p3} has the form

$$\|M\|\cdot\|c_{t3}\| = \|F_{t3}\|$$

$$F_{t3} = 6(\nu-1)^2\frac{\sin^2\gamma_t}{\gamma_t}\sum_{p=1}^{\infty}\frac{\sin^2\gamma_p}{\gamma_p}c_{p_2} + \frac{F_{t2}}{2\gamma_t} + 2\sum_{\substack{p=1\\p\neq t}}^{\infty}\frac{\cos^2\gamma_t-\cos^2\gamma_p}{\gamma_t^2-\gamma_p^2}\gamma_t\gamma_p \times \tag{2.3}$$

$$\times\left[1-\nu^2\frac{(\gamma_t+\gamma_p)^2}{(\gamma_t-\gamma_p)^2}\right]c_{p_2} - 2\nu\gamma_t^2\left(1+\nu-\frac{2}{3}\nu\cos^2\gamma_t\right)c_{t2} \qquad (t = 1,\,2,\,3,\,\ldots)$$

Here $||M||$ is the same matrix as in system (1.8). The system (2.3) is solved in the same manner as system (1.9), noting that $c_{2n,3}$ is the conjugate of $c_{2n-1,3}$ and setting

$$c_{2n-1,3}=\frac{k}{\mu}(y_{2n-1}-iy_{2n}) \qquad (n=1,\ 2,\ 3,\ \ldots) \tag{2.4}$$

Thereupon, it is possible to obtain the values (Table 4) of the first ten unknown y_j, as it is clear from (2.3) that with fourteen known values of x_j the order of the truncated y_j system will also be fourteen.

TABLE 4

	$j=$	1	2	3	4	5
	$\bar{y}_j=$	$-0.36003\cdot10^{-3}$	$-0.5072\cdot10^{-4}$	$0.190\cdot10^{-5}$	$0.9050\cdot10^{-5}$	$0.200\cdot10^{-5}$
$m=3$	$j=$	6	7	8	9	10
	$\bar{y}_j=$	$0.26\cdot10^{-6}$	$0.21\cdot10^{-6}$	$-0.5\cdot10^{-7}$	$0.11\cdot10^{-6}$	$-0.4\cdot10^{-7}$
	$j=$	1	2	3	4	5
	$\bar{y}_j=$	$-0.37035\cdot10^{-3}$	$-0.6225\cdot10^{-4}$	$-0.1661\cdot10^{-4}$	$0.1155\cdot10^{-4}$	$0.38\cdot10^{-6}$
$m=5$	$j=$	6	7	8	9	10
	$\bar{y}_j=$	$0.221\cdot10^{-5}$	$0.48\cdot10^{-6}$	$0.33\cdot10^{-6}$	$0.19\cdot10^{-6}$	$0.3\cdot10^{-7}$

To illustrate the rate of convergence of the process determining y_j, successive approximations of y_1, y_2, y_9 and y_{10} are shown (Table 5)

TABLE 5

	n	$y_1^{(n)}$	$y_2^{(n)}$	$y_9^{(n)}$	$y_{10}^{(n)}$
	4	$-0.36050\cdot10^{-3}$	$-0.4938\cdot10^{-4}$		
	6	$-0.35991\cdot10^{-3}$	$-0.50775\cdot10^{-4}$		
	8	$-0.35998\cdot10^{-3}$	$-0.50738\cdot10^{-4}$		
$m=3$	10	$-0.36000.10^{-3}$	$-0.50732\cdot10^{-4}$		
	12	$-0.360014\cdot10^{-3}$	$-0.50726\cdot10^{-4}$	$0.108\cdot10^{-6}$	$-0.49\cdot10^{-7}$
	14	$-0.360020\cdot10^{-3}$	$-0.50724\cdot10^{-4}$	$0.1075\cdot10^{-6}$	$-0.46\cdot10^{-7}$
	4	$-0.37288\cdot10^{-3}$	$-0.61675\cdot10^{-3}$		
	6	$-0.37072\cdot10^{-3}$	$-0.62075\cdot10^{-3}$		
	8	$-0.37037\cdot10^{-3}$	$-0.62262\cdot10^{-3}$		
$m=5$	10	$-0.37036\cdot10^{-3}$	$-0.62258\cdot10^{-3}$		
	12	$-0.370354\cdot10^{-3}$	$-0.62255\cdot10^{-3}$	$0.190\cdot10^{-6}$	$0.23\cdot10^{-7}$
	14	$-0.370352\cdot10^{-3}$	$-0.62253\cdot10^{-3}$	$0.191\cdot10^{-6}$	$0.27\cdot10^{-7}$

We may now determine the second approximations of the stress components. The convergence of the series in this step will now be checked. Substituting (2.2), (2.4) and the previously determined values of y_j into (1.1) and (1.3), we obtain, for $m = 3$,

$$\begin{aligned}\sigma_{n2}\,|_{\Gamma,\zeta=\pm1} = \pm k\{[-11.7504+0.714+0.498+0.080+0.062+\ldots]+[10.71918-\\ -0.1446-0.1832-0.1518-0.0456-0.0242-0.0138-\ldots]\}\,10^{-2}\approx\\ \approx\pm k\,(-0.1040+0.1016)=\mp k0.0024\end{aligned}$$

$$\begin{aligned}\tau_{nz2}\,|_{\Gamma,\zeta=0} = -k\{[1.3572+1.504-0.381+0.03-0.02+\ldots]+\\ +[-2.03747-0.6441+0.2492-0.0939+0.0359-0.013-0.004+\ldots]\}\,10^{-2}\approx\\ \approx -k\,(0.0249-0.0251)=k\,0.0002\end{aligned}$$

while for $m = 5$, we shall have

$$\sigma_{n2}|_{\Gamma,\,\zeta=\pm1} = \pm k\{[-12.3382 - 1.1650 + 0.290 + 0.242 + 0.126 + \ldots] + [11.3542 + 1.4340 + 0.0244 - 0.0850 - 0.0664 - 0.0432 - 0.0254 - \ldots]\}\,10^{-2} \approx \pm k(-0.1285 + 0.1259) = \mp k0.0026$$

$$\tau_{nz2}|_{\Gamma,\,\zeta=0} = -k\{[1.774 + 1.221 - 0.894 + 0.40 - 0.15 + \ldots] + [-2.384 - 0.261 + 0.419 - 0.236 + 0.125 - 0.06 + 0.03 + \ldots]\}\,10^{-2} \approx -k(0.0235 - 0.0237) = k0.0002$$

Comparing the results thus obtained with the values given in (1.15) for the stresses σ_n and τ_{nz} on the boundary Γ, we find that convergence of the second approximation is also satisfactory.

Now consider the stress $\sigma_s|_\Gamma$. In a manner similar to that of section 1, it is readily shown that $\sigma_s|_{\Gamma,\,\zeta=\pm1}$ may also be obtained for the second approximation without solving the infinite system of equations to determine c_{p3}. Thus we obtain

$$\sigma_{s2}|_{\Gamma,\zeta=\pm1} = \mp\mu\frac{3\nu-1}{\nu}\sum_{p=1,3,\ldots}\mathrm{Re}\left\{\left[6(\nu-1)\frac{\sin^2\gamma_p}{\gamma_p} + 2\gamma_p n_p(1)\right]c_{p_2}\right\} \qquad (2.5)$$

Calculations utilizing (2.5) yield, for $m = 3$,

$$\sigma_{s2}\Big|_{\Gamma,\,\zeta=\pm1} = \mp k\frac{8}{3}[5.35959 - 0.0723 - 0.0916 - 0.0759 - 0.0228 - 0.0121 - 0.0069 - \ldots]\cdot10^{-2} \approx \mp k\cdot0.135$$

For $m = 5$, we obtain

$$\sigma_{s2}\Big|_{\Gamma,\,\zeta=\pm1} = \mp k\frac{8}{3}[5.6771 + 0.7170 + 0.0122 - 0.0425 - 0.0332 - 0.0216 - 0.0127 - \ldots]\cdot10^{-2} \approx \mp k\cdot0.168$$

3. The third step of the construction of the asymptotic expansion deals with the determination of $\psi_2(n)$ and c_{p4}. The boundary conditions for $\psi_2(n)$ are given by

$$\frac{1}{3}\left[2\nu\frac{\partial^2\psi_2}{\partial n^2} + (\nu-1)\frac{\partial\psi_2}{\partial n}\right]_{n=0} = (\nu-1)\sum_{p=1}^{\infty}\frac{\sin^2\gamma_p}{\gamma_p}\left(c_{p_3} + \frac{1}{2\gamma_p}c_{p_2}\right), \qquad \frac{\partial\Delta\psi_2}{\partial n}\Big|_{n=0} = 0 \qquad (3.1)$$

From (3.1), we have

$$\psi_2 = -\frac{3(\nu-1)}{\nu+1}\sum_{p=1}^{\infty}\frac{\sin^2\gamma_p}{\gamma_p}\left(c_{p_3} + \frac{1}{2\gamma_p}c_{p_2}\right)\ln(n+1) \qquad (3.2)$$

To determine c_{p4}, we have the infinite system of equations with the previously given matrix, but, as previously discussed, it is easily shown that the third approximation of $\sigma_s|_{\Gamma,\,\zeta=\pm1}$ may be obtained without the determination of c_{p4}. Thus, we obtain

$$\sigma_{s3}|_{\Gamma,\zeta=\pm1} = \mp\mu\frac{3\nu-1}{\nu}\sum_{p=1,3,\ldots}\mathrm{Re}\left\{\left[6(\nu-1)\frac{\sin^2\gamma_p}{\gamma_p} + 2\gamma_p n_p(1)\right]\left(c_{p_3} + \frac{1}{2\gamma_p}c_{p_2}\right)\right\} \qquad (3.3)$$

Calculations utilizing (3.3) yield for $m = 3$,

$$\sigma_{s3}|_{\Gamma,\,\zeta=\pm 1} = \mp k\,{}^{8}/_{3}\{[-1.2423+0.0335+0.0170+0.0021+0.0013+\ldots]+$$
$$+{}^{1}/_{2}[1.64360+0.00538-0.0178-0.0034-0.0014-0.00065-\ldots]\}\,10^{-2}\approx$$
$$\approx \pm k\cdot 0.0100$$

For $m = 5$, we obtain

$$\sigma_{s3}|_{\Gamma,\,\zeta=\pm 1} = \mp k\,{}^{8}/_{3}\{[-1.3005-0.06460+0.0090+0.0061+0.0026+\ldots]+$$
$$+{}^{1}/_{2}[1.7137+0.1335+0.00629-0.0022-0.0018-0.0010-0.0006-\ldots]\}\,10^{-2}\approx$$
$$\approx \pm k\,0.0113$$

We now present a three-term approximation of the asymptotic expansions of $\sigma_s|\Gamma,\ \zeta = \pm 1$

$$\sigma_s|_{\Gamma,\zeta=\pm 1} = \pm k\,[-0.4667\,\lambda - 0.135\,\lambda^2 + 0.0100\lambda^3 + \ldots] \quad \text{for} \quad m=3 \tag{3.4}$$
$$\sigma_s|_{\Gamma,\,\zeta=\pm 1} = \pm k\,[-0.2381\,\lambda - 0.168\,\lambda^2 + 0.0113\,\lambda^3 + \ldots] \quad \text{for} \quad m=5 \tag{3.5}$$

From (3.4) and (3.5) it is clear that even for $\lambda = 2$ (i.e. when the plate thickness is twice the diameter of the opening the third term in the expansion represents only 5% to 8% of the sum of the first two terms. As a result, we can recommend for $\lambda < 2$ that the determination of the stress concentration factor be based on the first two terms of the expansion in powers of λ.

It is readily seen from (3.4) and (3.5) that, for $\lambda = 0.1$, the sum of the next two terms is equal to 3% to 7% of the first term. Thus, for $\lambda < 0.1$, the stress concentration factor for σ_s may be obtained from the following expression,

$$\sigma_s|_{\Gamma,\zeta=\pm 1} \approx \pm k\lambda\left[-\frac{3}{m+2} + \frac{\nu-1}{2\nu}\,\frac{m-1}{m+2}\right] \tag{3.6}$$

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